An Iterative Solution of the N/D Equation*

MYRON BANDER
Stanford Linear Accelerator Center, Stanford University, Stanford, California
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An iterative scheme is presented for solving the N/D equations in the case where the left-hand cut consists entirely as a sum of poles. At no step is recourse to a matrix inversion of an algebraic system required. A scheme for approximating arbitrary cuts by sequences of poles is also presented.

I. INTRODUCTION

In many applications of the partial wave dispersion relations one is faced with nonlinear integral equations which may be linearized by the N/D method.[1] A left-hand cut discontinuity for the amplitude is assumed to be of some given form, and likewise some assumptions are made about the unitarity on the right-hand cut; usually elastic unitarity is assumed or the inelastic contributions are approximated by several two-body channels.[2] The resulting integral equations are either of the Fredholm type or have a kernel which is singular due only to infinite integration ranges. This difficulty is usually overcome by the introduction of a cut-off and the resulting equations lend themselves to standard numerical solutions, which in practice usually require the use of a high-speed computer.

Often a more drastic assumption is made about the left-hand cut, and it is replaced by a series of poles.[3] The integral equations are then reducible to a linear algebraic system. If the number of poles is large, we are then faced with the inversion of matrices of large order, which, even if the problems are solved with the aid of computers, it is the matrix inversion which consumes most of the time.

In this article an iterative method is presented by which we may go from an n-pole to an $(n+1)$-pole problem directly without ever introducing the necessity of matrix inversion. This method is also applicable to the following case. Suppose that we have the solution for a certain left-hand cut, then we may immediately obtain the solution for a new cut which is equal to the old one plus a finite number of pole terms. This method is possible due to the fact that we have a freedom of choosing arbitrarily a subtraction point where the D function is normalized to unity.

Section II is devoted to the derivation of the iteration scheme starting from a general left-hand cut, and adding an arbitrary number of poles. In Sec. III this scheme is specialized to the left-hand cut consisting entirely of poles. In Sec. IV a discussion is given on how to obtain a sequence of poles approximation to any cut. All the results are given for a single-channel case, although they may easily be generalized to a many-channel problem.

II. ITERATION ON THE RESOLVENT KERNELS

In the N/D method we write the amplitude as a ratio of two functions N and D which have discontinuities, respectively, on the left or, respectively, right-hand cuts only. We choose to write an integral equation for N and express D in terms of N.

$$N(x;x_p) = B(x) + \frac{1}{\pi}\int \frac{\rho(z)N(z;x_p)}{(z-x)(z-x_p)} \times [B(z)(z-x_p) - B(x)(x-x_p)]\,dz, \quad \text{(II 1)}$$

$$D(x;x_p) = 1 - \frac{x-x_p}{\pi}\int \frac{\rho(z)N(z;x_p)}{(z-x)(z-x_p)}\,dz, \quad \text{(II 2)}$$

where $B(z)$ is a known function with only left-hand discontinuities and $\rho(z)$ is a phase space factor, which is a known kinematical function. The integrals in the above equation run over only the positive real axis from the start of the elastic cut. x_p is an arbitrary point at which we may normalize $D(x;x_p)$ to unity. The ratio $N(x;x_p)/D(x;x_p)$ is independent of x_p,[4] and in terms of the function $\rho(x)$ equals $\exp[i\delta(x)]\sin\delta(x)/\rho(x)$.

Instead of Eq. (II 1) let us consider an equation with the inhomogeneous term $B(x)$ replaced by an arbitrary function, $f(x)$.

$$N_f(x;x_p) = f(x) + \frac{1}{\pi}\int \frac{\rho(z)N_f(z;x_p)}{(z-x)(z-x_p)} \times [B(z)(z-x_p) - B(x)(x-x_p)]\,dz. \quad \text{(II 3)}$$

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[1] G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960).
[2] J. Bjorken, Phys. Rev. Letters 4, 473 (1960).
[3] See for example: H. P. Noyes and D. Y. Wong, Phys. Rev. Letters 3, 191 (1959); W. Frazer and J. Fulco, *ibid.* 2, 365 (1959).
[4] This result has been obtained by A. W. Martin (unpublished).

We assume that the above integral equation has a solution; namely, there exists a resolvent kernel, $G(x, y; x_p)$ such that

$$N_f(x; x_p) = \int G(x, y; x_p) f(y)\, dy. \qquad \text{(II 4)}$$

The first step in this iteration procedure is to find the transformation between a resolvent kernel $G(x, y; x_p)$ for a subtraction point at x_p and the resolvent kernel $G(x, y; x_q)$ for another subtraction point x_q. Adding and subtracting to Eq. (II 3)

$$-\frac{1}{\pi}\int \frac{\rho(z)N_f(z; x_p)}{(z - x)(z - x_q)} B(x)(x - x_q)\, dz, \qquad \text{(II 5)}$$

we obtain:

$$N_f(x; x_p) = f(x) + \frac{1}{\pi}\int \frac{\rho(z)N_f(z, x_p)}{(z - x)(z - x_q)} \times [B(z)(z - x_q) - B(x)(x - x_q)]\, dz + \frac{B(x)}{\pi}(x_p - x_q)\int \frac{\rho(z)N_f(z; x_p)}{(z - x_q)(z - x_p)}\, dz. \qquad \text{(II 6)}$$

As the last term of Eq. (II 6) is $CB(x)$, where C is independent of x, we may write

$$N_f(x; x_p) = \int G(x, y; x_q)[f(y) + CB(y)]\, dy, \qquad \text{(II 7)}$$

where

$$C = \frac{x_p - x_q}{\pi}\int \frac{\rho(z)N_f(z; x_p)}{(z - x_q)(z - x_p)}\, dz. \qquad \text{(II 8)}$$

Substituting Eq. (II 7) into Eq. (II 8) we obtain C:

$$C = \frac{\int H(y; x_p, x_q) f(y)\, dy}{1 - \int H(y; x_p, x_q) B(y)\, dy}, \qquad \text{(II 9)}$$

$$H(y; x_p, x_q) = \frac{x_p - x_q}{\pi} \times \int \frac{\rho(z)}{(z - x_p)(z - x_q)} G(z, y; x_q)\, dz. \qquad \text{(II 10)}$$

The sought for relation between $G(x, y; x_p)$ and $G(x, y; x_q)$ is

$$G(x, y; x_p) = G(x, y; x_q) + \frac{\left[\int G(x, y'; x_q)B(y')\, dy'\right] H(y; x_p, x_q)}{1 - \int H(y'; x_p, x_q)B(y')\, dy'}. \qquad \text{(II 11)}$$

Now the iteration scheme may be outlined. Suppose we add to $B(x)$ an extra pole,

$$\tilde{B}(x) = B(x) + \gamma_1/(x - x_1). \qquad \text{(II 12)}$$

Equation (II 3) for an arbitrary subtraction point x_p becomes

$$\tilde{N}_f(x; x_p) = f(x) + \frac{1}{\pi}\int \frac{\rho(z)}{(z - x)(z - x_p)} \times \left[B(z)(z - x_p) - B(x)(x - x_p) + \gamma_1\left(\frac{z - x_p}{z - x_1} - \frac{x - x_p}{x - x_1}\right)\right]\tilde{N}_f(z; x_p)\, dz. \qquad \text{(II 13)}$$

If we choose $x_p = x_1$ the kernel of Eq. (II 13) reduces to the kernel of Eq. (II 3). Thus

$$\tilde{G}(x, y; x_1) = G(x, y; x_1). \qquad \text{(II 14)}$$

The iteration scheme we propose is as follows: Given $G(x, y; x_p)$ we obtain via Eq. (II 11) $G(x, y; x_1)$ which equals $\tilde{G}(x, y; x_1)$ which we may again via Eq. (II 11) transform to another subtraction point x_2 and add a pole at x_2. If we denote by $G^{(n)}(x, y; x_p)$ the resolvent kernel with n extra poles located at x_i, with residues γ_i, $i = 1, \cdots, n$, the scheme may be outlined as

$$G^{(n)}(x, y; x_n) \to G^{(n)}(x, y; x_{n+1}) = G^{(n+1)}(x, y; x_{n+1}) \to G^{(n+1)}(x, y; x_{n+2}) = G^{(n+2)}(x, y; x_{n+2}) \to \cdots, \qquad \text{(II 15)}$$

where the arrows indicate application of Eq. (II 11) with appropriate $B(x)$, namely in going from $G^{(1)}(x, y; x_i)$ to $G^{l}(x, y; x_{l+1})$ the $B(x)$ that enters into Eq. (II 11) is

$$B(x) + \sum_{i=1}^{l}\frac{\gamma_i}{x - x_i}. \qquad \text{(II 16)}$$

Thus, each step of the iteration procedure is reduced to quadratures and at no point do we encounter a problem of matrix inversion.

Before proceeding further we show that the amplitude $N(x; x_p)/D(x, x_p)$ is independent of x_p. By definition

$$N(x; x_p) = \int G(x, y; x_p)B(y)\, dy, \qquad \text{(II 17)}$$

$$D(x_p; x_q) = 1 - \int H(y; x_p, x_q)B(y)\, dy. \qquad \text{(II 18)}$$

From Eq. (II 11) we obtain

$$N(x; x_p) = N(x; x_q)/D(x_p; x_q). \qquad \text{(II 19)}$$

As $D(x, x_q)/D(x_p, x_q)$ is one for $x = x_p$ and its imaginary part equals $-\rho(x)N(x; x_p)$, it satisfies

Eq. (II 2) with the subtraction point at x_p, and thus $N(x; x_p)/D(x; x_p) = N(x; x_q)/D(x; x_q)$.[4]

III. ITERATION OF THE n-POLE PROBLEM

If $B^n(x)$ consists entirely as a sum of pole terms

$$B^n(x) = \sum_{i=1}^{n} \frac{\gamma_i}{x - x_i}, \tag{III 1}$$

the iteration scheme and especially Eq. (II 11) take on a much simpler form. In this case Eq. (II 3) may be reduced to

$$\begin{aligned} N_f(x; x_p) &= f(x) - \frac{1}{\pi} \sum_{i=1}^{n} \frac{\gamma_i(x_i - x_p)}{x - x_i} \\ &\quad \times \int \frac{\rho(z) N_f(z; x_p)}{(z - x_p)(z - x_i)}\, dz, \\ &= f(x) - \sum_{i=1}^{n} \frac{n_i(x_p)}{x - x_i}, \end{aligned} \tag{III 2}$$

where the $n_i(x_p)$ are the solutions of a linear algebraic system;

$$\begin{aligned} n_i(x_p) &= \frac{1}{\pi} \gamma_i(x - x_p) \int \frac{\rho(z) f(z)}{(z - x_p)(z - x_i)}\, dz \\ &\quad - \frac{1}{\pi} \gamma_i(x_i - x_p) \int \frac{\rho(z)}{(z - x_p)(z - x_i)} \sum_{i=1}^{n} \frac{n_i(x_p)}{z - x_i}\, dz. \end{aligned} \tag{III 3}$$

There exists a resolvent matrix $G_{ij}^{(n)}(x_p)$ such that

$$\begin{aligned} n_i(x_p) &= \sum_{j=1}^{n} G_{ij}(x_p)\gamma_j \frac{(x_j - x_p)}{\pi} \\ &\quad \times \int \frac{\rho(z) f(z)}{(z - x_p)(z - x_j)}\, dz. \end{aligned} \tag{III 4}$$

Going through a procedure analogous to that of Sec. II we derive a transformation between $G_{ij}(x_p)$ and $G_{ij}(x_q)$

$$G_{ij}(x_p) = G_{ij}(x_q) + \frac{G_{im}(x_q)\gamma_m K(x_p, x_q)_l G_{lj}(x_q)}{1 - K(x_p, x_q)_l G_{lk}(x_q)\gamma_k}, \tag{III 5}$$

where

$$K(x_p, x_q)_l = \frac{1}{\pi}(x_p - x_q) \int \frac{\rho(z)\, dz}{(z - x_p)(z - x_q)(z - x_l)}, \tag{III 6}$$

and the summation convention has been adopted. The iterative scheme analogous to Eq. (II 15) is:

$$\begin{aligned} &G^{(k)}(x_p) \to G^{(k)}(x_{k+1}), \\ &G^{(k+1)}(x_{k+1}) = G^{(k)}(x_{k+1}) \oplus 1, \\ &G^{(k+1)}(x_{k+1}) \to G^{(k+1)}(x_{k+2}), \\ &\quad \vdots \end{aligned} \tag{III 7}$$

where the arrow indicates an application of Eq. (III 5), and the notation $A \oplus 1$ means that if A is a $k \times k$ matrix, $A \oplus 1$ is a $(k + 1) \times (k + 1)$ matrix with

$$\begin{aligned} (A \oplus 1)_{ij} &= A_{ij}, \quad i, j \leq k, \\ &= 0, \quad i = k + 1, \quad j \leq k \\ &\qquad \text{or} \quad j = k + 1, \quad i \leq k, \\ &= 1; \quad i = j = k + 1. \end{aligned} \tag{III 8}$$

IV. APPROXIMATION OF CUTS BY SEQUENCES OF POLES[5]

The functions $B(x)$ appearing in the previous equations are analytic functions cut along curves in the complex plane which are disjointed from the right-hand unitarity cut. In the equal mass case it is generally a single cut running along the negative real axis. More generally it may have additional cuts along finite curves.[6] In most applications $B(x)$ has at least one cut extending to infinity. To discuss any approximation technique it is convenient to make a change of variables such as

$$x = a/(u + b), \tag{IV 1}$$

which makes all integration ranges finite. Such a change leaves the integral equations (II 1)–(II 3) of the same form. Now the problem is to approximate the transformed kernel $B(u)$ by a sequence of pole. The general expression for $B(u)$ will be of the form,

$$B(u) = (u + b) \sum_i \int_{c_i} \frac{\omega_i(z)}{z - u}\, dz. \tag{IV 2}$$

Baker, Gammel, and Wills[7] have suggested a scheme using the Padé approximants. Their scheme consists of expanding $B(u)$ as a power series in u^{-n}, and writing each partial sum as a Padé approximant. These authors have shown that the sequence of approximants converges to the desired function under very general conditions. The cuts of Eq.

[5] All the results of this section are based on the analytic theory of continued fractions and on the theory of orthogonal polynomials. General references for this section are:
H. S. Wall, *Analytic Theory of Continued Fractions* (D. Van Nostrand, Inc., Princeton, New Jersey, 1948); J. A. Shohat and J. D. Tamarkin, *The Problem of Moments* (American Mathematical Society, New York, New York, 1943), Chaps. I and II; O. Perron, *Die Lehre von den Kettenbrüchen* (B. G. Teubner, Leipzig and Berlin, 1913); G. Szegö, *Orthogonal Polynomials* (American Mathematical Society, New York, New York, 1959), specifically Chaps. II, III, and XVI.

[6] In problems involving fermions it is convenient to work in the energy instead of energy-squared plane, where the unitarity cut runs along the real axis excluding an interval about zero, and the function $B(x)$ has a cut along the imaginary axis.

[7] G. A. Baker, J. L. Gammel, and J. G. Wills, J. Math. Anal. Appls. 2, 21 (1961).

(IV 2) are approximated by a sequence of poles whose positions approach the cuts themselves.

We shall present a method, which, although much more tedious, has the advantage that it makes few assumptions on the function $B(u)$ and likewise shows that the poles of the approximating sequences are not on the unitarity cut. In actual practice the scheme of Ref. 7 is strongly recommended. Although both methods are derived from the analytic structure of continuous fractions, the exact relation between them is not investigated.

The assumptions we shall make on the function $B(u)$ is that the contours c_i are rectifiable; that there exists a convex region containing each c_i,[8] such that it does not intersect the unitarity cut, and that the functions $\omega_i(z)$ are of bounded variation. Under these assumptions, let us rewrite Eq. (IV 2) as

$$B(u) = \sum_i \eta_i \int_{c_i} \frac{v_i(z)\ |dz|}{u - z}, \qquad \text{(IV 3)}$$

where $|dz|$ is the arc length along the curves c_i, and the factors η_i are chosen so that $v_i(z)$ is a nonnegative real function.

Now we are in a position to approximate each term in Eq. (IV 3) by a sequence of poles. As $v_i(z)$ is a nonnegative function, we may define a set of orthogonal polynomials with $v_i(z)$ as a weight function, i.e.,

$$\int_{c_i} p_n^{(i)*}(u) p_m^{(i)}(u) v_i(u)\ |du| = \delta_{mn}. \qquad \text{(IV 4)}$$

The coefficients of the $p_n^{(i)}(u)$ are real, and the zeros lie in the least convex region containing the curve c_i, and approach c_i for n sufficiently large. The zeros of $p_n^{(i)}(u)$ do not lie along the end points of c_i for any finite n.

Let

$$\int_{c_i} v_i(u) u^{*n} u^m\ |du| = c_{mn} \qquad \text{(IV 5)}$$

and

$$Q_n^{(i)}(x) = \int_{c_i} \frac{p_n^{(i)}(u) - p_n^{(i)}(x)}{u - x} v_i(u)\ |du|.$$

The $Q_n^{(i)}(x)$ are polynomials of degree $n - 1$. By a theorem due to Markoff we have our result:

$$\int \frac{v_i(u)\ |du|}{u - x} = -\lim_{n\to\infty} c_{00}^2 (c_{00}c_{11} - c_{10}c_{01})^{-\frac{1}{2}} \frac{Q_n^{(i)}(x)}{P_n^{(i)}(x)},$$

which by partial fractions may be expressed as a sum of poles.

[8] The end of the transformed unitarity cut may coincide with the end of one of the cuts. This will cause no difficulty.